MATHEMATICS

Chapter 5: COMPLEX NUMBERS & QUADRATIC EQUATIONS

COMPLEX NUMBERS & QUADRATIC EQUATIONS

Some Important Results

1. Solution of $x^2 + 1 = 0$ with the property $i^2 = -1$ is called the imaginary unit.
2. Square root of a negative real number is called an imaginary number.
3. If a and b are positive real numbers, then $\sqrt{-a} \times \sqrt{-b} = -\sqrt{ab}$
4. If a is a positive real number, then we have $\sqrt{-a} = i\sqrt{a}$.
5. Powers of i

   $i = \sqrt{-1}$;

   $i^2 = -1$;

   $i^3 = -i$

   $i^4 = 1$
6. If n>4, then $i^{-n} = \frac{1}{i^n} = \frac{1}{k}$ where k is the reminder when n is divided by 4.
7. We have $i^o = 1$.
8. A number in the form a + ib, where a and b are real numbers, is said to be a complex number.
9. In complex number z = a + ib, a is the real part, denoted by Re z and b is the imaginary part denoted by Im z of the complex number z.
10. $\sqrt{-1}$ =i is called iota, which is a complex number.
11. The modulus of a complex number z = a + ib denoted by |z| is defined to be a non-negative real number $\sqrt{a \quad + b^2}$, i.e. $|z| = \sqrt{a^2 \quad b}$.
12. For any non-zero complex number z = a + ib (a ≠ 0, b ≠ 0), there exists a complex number $\frac{a}{a^2+b^2} + i\frac{(-b)}{a^2 \quad b^2}$, denoted by $\frac{1}{z}$ or z^{-1}, called the multiplicative inverse of z such that

   $$(a+ib)\times\left(\frac{a}{a^2+b^2} + i\frac{(-b)}{a^2+b^2}\right) = 1 + i0 = 1.$$
13. Conjugate of a complex number z = a + ib, denoted as z, is the complex number a − ib.

14. The number z = r(cos θ +isin θ) is the polar form of the complex number z = a + ib.

Here $r = \sqrt{a^2 + b^2}$ is called the modulus of z $\theta = \tan^{-1}\left(\frac{b}{a}\right)$ and is called the argument or amplitude of z, which is denoted by arg z.

15. The value of θ such that $-\pi < \theta \leq \pi$ called principal argument of z.

16. The Eulerian form of z is $z = re^{i\theta}$, where $e^{i\theta} = \cos\theta + i\sin\theta$

17. The plane having a complex number assigned to each of its points is called the Complex plane orArgand plane.

18. Let $a_0, a_1, a_2, \ldots$ be real numbers and x is a real variable. Then, the real polynomial of a real variable with real coefficients is given as

$f(x) = a_0 + a_1x + a_2x^2 + \ldots. a_nx^n$

19. Let $a_0, a_1, a_2, \ldots$ be complex numbers and x is a complex variable. Then, the real polynomial of a complex variable with complex coefficients is given as

$f(x) = a_0 + a_1x + a_2x^2 + \ldots. a_nx^n$

20. A polynomial $f(x) = a_0 + a_1x + a_2x^2 + \ldots. a_nx^n$ is a polynomial of degree n.

21. Polynomial of second degree is called a quadratic polynomial.

22. Polynomials of degree 3 and 4 are known as cubic and biquadratic polynomials.

23. If f(x) is a polynomial, then f(x) = 0 is called a polynomial equation.

24. If f(x) is a quadratic polynomial, then f(x) = 0 is called a quadratic equation.

25. The general form of a quadratic equation is $ax^2 + bx + c = 0$, $a \neq 0$.

26. The values of the variable satisfying a given equation are called its roots.

27. A quadratic equation cannot have more than two roots.

28. Fundamental Theorem of Algebra states that 'A polynomial equation of degree n has n roots.'

Top Concepts

1. **Addition of two complex numbers**: If $z_1 = a + ib$ and $z_2 = c + id$ be any two complex numbers, then the sum
$z_1 + z_2 = (a + c) + i(b + d)$.

2. Sum of two complex numbers is also a complex number. This is known as the closure property.

3. The addition of complex numbers satisfy the following properties:

   i. Addition of complex numbers satisfies the commutative law. For any two complex numbers z_1 and z_2, $z_1 + z_2 = z_2 + z_1$.

   ii. Addition of complex numbers satisfies associative law for any three complex numbers z_1, z_2, z_3, $(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3)$.

   iii. There exists a complex number 0 + i0 or 0, called the additive identity or the zero complex number, such that for every complex number z,
   $z + 0 = 0 + z = z$.

   iv. To every complex number $z = a + ib$, there exists another complex number $-z = -a + i(-b)$ called the additive inverse of z.
   $z+(-z)=(-z)+z=0$

4. **Difference of two complex numbers:** Given any two complex numbers If $z_1 = a + ib$ and $z_2 = c + id$ the difference $z_1 - z_2$ is given by
$z_1 - z_2 = z_1 + (-z_2) = (a - c) + i(b - d)$.

5. **Multiplication of two complex numbers** Let $z_1 = a + ib$ and $z_2 = c + id$ be any two

complex numbers. Then, the product $z_1 z_2$ is defined as follows:

$z_1 z_2 = (ac - bd) + i(ad + bc)$

6. **Properties of multiplication of complex numbers**: Product of two complex numbers is a complexnumber; the product $z_1 z_2$ is a complex number for all complex numbers z_1 and z_2.
   i. Product of complex numbers is commutative, i.e. for any two complex numbers z_1 and z_2,$z_1 z_2 = z_2 z_1$
   ii. Product of complex numbers is associative, i.e. for any three complex numbers z_1, z_2, z_3,$(z_1 z_2) z_3 = z_1 (z_2 z_3)$.
   iii. There exists a complex number 1 + i0 (denoted as 1), called the multiplicative identitysuch that z.1 = z for every complex number z.
   iv. For every non-zero complex number, z = a + ib or a + bi (a ≠ 0, b ≠ 0), there is a complex number $\frac{a}{a^2+b^2} + i\frac{-b}{a^2\ \ b^2}$ called the multiplicative inverse of z such that $z \times \frac{1}{z} = 1$
   v. distributive law: For any three complex numbers z_1, z_2, z_3,
      a. $z_1 (z_2 + z_3) = z_1.z_2 + z_1.z_3$
      b. $(z_1 + z_2) z_3 = z_1.z_3 + z_2.z_3$

7. **Division of two complex numbers**: Given any two complex numbers $z_1 = a + ib$ and $z_2 = c + id$, where $z_2 \neq 0$, the quotient $\frac{z_1}{z_2}$ is defined by

$$\frac{z_1}{z_2} = z_1.\frac{1}{} = \frac{ac+bd}{c^2+d^2} + i\frac{bc-ad}{c^2\ \ d^2}.$$

8. Identities for complex numbers

i. $(z_1 + z_2)^2 = z_1^2 + z_2^2 + 2z_1.z_2$, for all complex numbers z_1 and z_2.
ii. $(z_1 - z_2)^2 = z_1^2 - 2z_1z_2 + z_2^2$
iii. $(z_1 + z_2)^3 = z_1^3 + 3z_1^2z_2 + 3z_1z_2^2 + z_2^3$
iv. $(z_1 - z_2)^3 = z_1^3 - 3z_1^2z_2 + 3z_1z_2^2 - z_2^3$
v. $z_1^2 - z_2^2 = (z_1 + z_2)(z_1 - z_2)$

9. Properties of modulus and conjugate of complex numbers

For any two complex numbers z_1 and z_2,

i. $|z_1 z_2| = |z_1||z_2|$

ii. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z|}$ provided $|z_2| \neq 0$

iii. $\overline{z_1 z_2} = \overline{z_1}\,\overline{z_2}$

iv. $\overline{z_1 \pm z_2} = \overline{z_1} \pm \overline{z_2}$

v. $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ provided $z_2 \neq 0$

vi. $\overline{(\overline{z})} = z$

vii. $z + \overline{z} = 2\text{Re}(z)$

viii. $z - \overline{z} = 2i\,\text{Im}(z)$

ix. $z = \overline{z} \Leftrightarrow z$ is purely real

x. $z + \overline{z} = 0 \Rightarrow z$ is purely imaginary

xi. $z\overline{z} = [\text{Re}(z)]^2 + [\text{Im}(z)]^2$

10. The order of a relation is not defined in complex numbers. Hence there is no meaning in $z_1 >$ z.

11. Two complex numbers z_1 and z_2 are equal iff Re (z_1) = Re (z_2) and Im (z_1) Im (z_2).

12. The sum and product of two complex numbers are real if and only if they are conjugate of each other.

13. For any integer k, $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$, $i^{4k+3} = -i$. $\sqrt{a} \times \sqrt{b} \neq \sqrt{ab}$ when a<0 and b<0.

14. The polar form of the complex number $z = x + iy$ is $r(\cos\theta + i\sin\theta)$, where r is the modulus of z and θ is known as the argument of z.

15. For a quadratic equation $ax^2 + bx + c = 0$ with real coefficients a, b and c and $a \neq 0$. If the discriminant $D = b^2 - 4ac \geq 0$, then the equation has two real roots given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{or} \quad x = \frac{-b}{2a}.$$

16. Roots of the quadratic equation $ax^2 + bx + c = 0$, where a, b and $c \in R$, $a \neq 0$, when discriminant $b^2 - 4ac < 0$, are imaginary given by

$$x = \frac{-b \pm \sqrt{4ac - b^2}i}{2a}.$$

17. Complex roots occur in pairs.

18. If a, b and c are rational numbers and b^2 - 4ac is positive and a perfect square, then $\sqrt{b^2 - 4ac}$ is a rational number and hence the roots are rational and unequal.

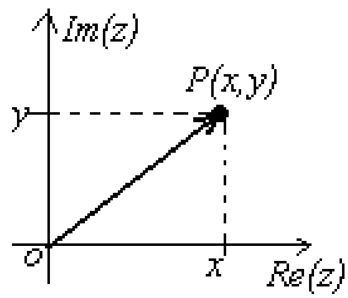
19. If b^2 - 4ac = 0, then the roots of the quadratic equation are real and equal.

20. If b^2 - 4ac = 0 but it is not a perfect square, then the roots of the quadratic equation are irrational and unequal.

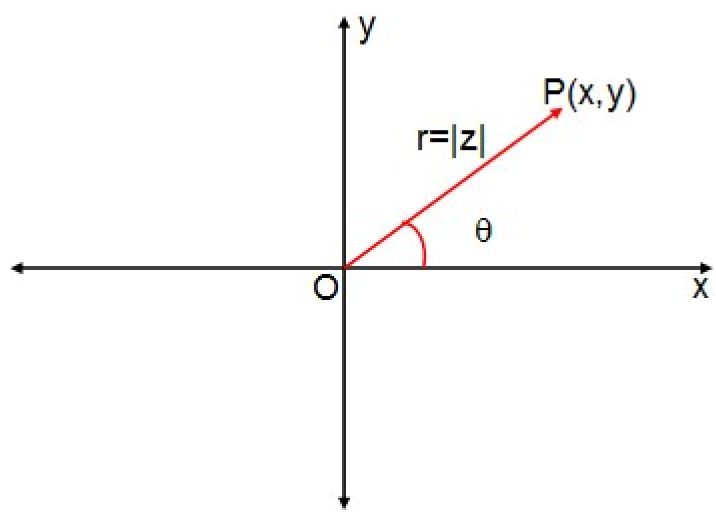
21. Irrational roots occur in pairs.

22. A polynomial equation of n degree has n roots. These n roots could be real or complex.

23. Complex numbers are represented in the Argand plane with X-axis being real and Y-axis being imaginary.



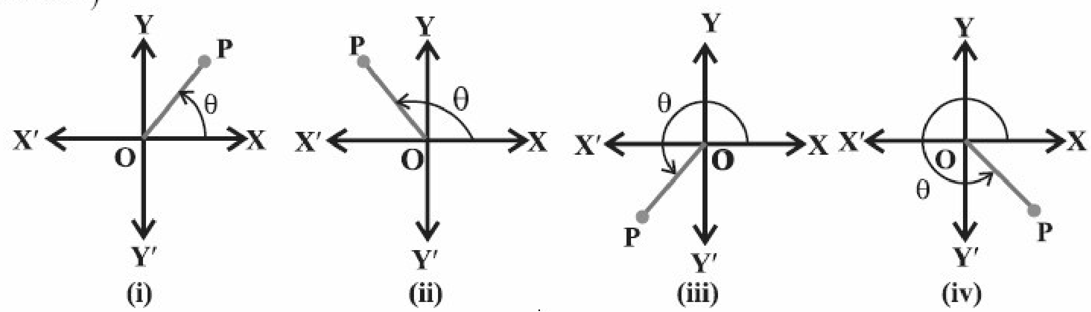
24. Representation of complex number z = x + iy in the Argand plane.



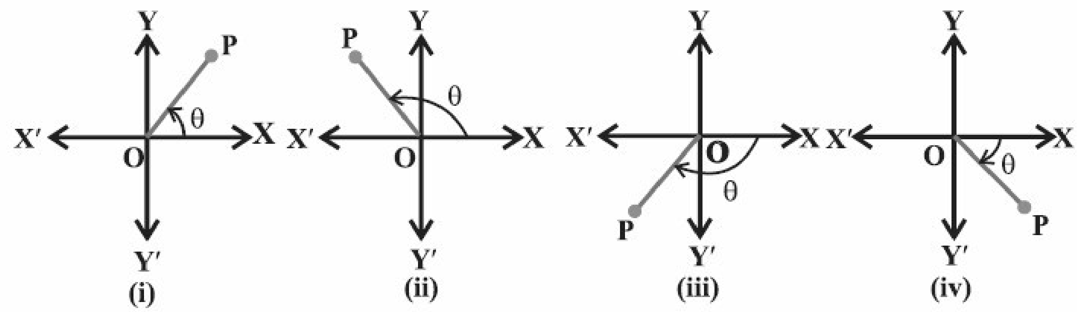
25. Multiplication of a complex number by i results in rotating the vector joining the origin to the pointrepresenting z through a right angle.

26. Argument θ of the complex number z can take any value in the interval [0, 2π). Different orientations of z are as follows

$(0 \leq \theta < 2\pi)$



$(-\pi < \theta \leq \pi)$



MIND MAP : LEARNING MADE SIMPLE CHAPTER - 5

Complex Number & Quadratic Equations

Definition of Complex Numbers

A number of the form a+*ib*, where a,b ∈R and $i=\sqrt{-1}$ is called a complex number and denoted by 'z'.

z= a+*ib* → *ib*: Imaginary part; a: Real part

Conjugate of a complex number: For a given complex number z=a+*ib*, its conjugate is defined as $\bar{z}$= a – *ib*

Algebra of Complex Numbers

Let: z_1= a+*ib* and z_1= c+*id* be two complex numbers, where a,b,c,d ∈ R and $i=\sqrt{-1}$

1. Addition: $z_1+z_2=(a+ib)+(c+id)=(a+c)+i(b+d)$

2. Subtraction: $z_1-z_2=(a+ib)-(c+id)=(a-c)+(b-d)i$

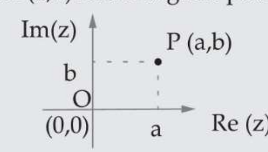
3. Multiplication: $z_1 \cdot z_2=(a+ib)\,(c+id)$
$=a(c+id)+ib(c+id)$
$=(ac-bd)+(ad+bc)i$
$(\because i^2=-1)$

4. Division: $\frac{z_1}{z_2}=\frac{a+ib}{c+id}=\frac{a+ib}{c+id}\cdot\frac{c-id}{c-id}$
$=\left(\frac{ac+bd}{c^2+d^2}\right)+\left(\frac{bc-ad}{c^2+d^2}\right)i$

Note: If a+*ib* = c+*id* ⇔ a = c & b = d

Argand Plane

A complex number z=a+ib can be represented by a unique point P(a,b) in the argand plane



z=a+ib is represented by a point P (a,b)

Powers of 'i'

$i=\sqrt{-1}, i^2=-1$

In general, $i^{4k+r}=\begin{cases}1; r=0\\ i; r=1\\ -1; r=2\\ -i; r=3\end{cases}$

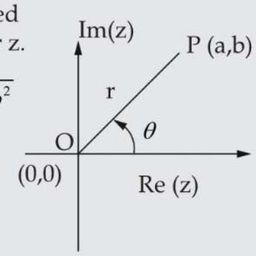
Modulus & Argument of Complex Number

If z= a+ib is a complex number

(i)Distance of z from origin is called as modulus of complex number z.

It is denoted by $r=|z|=\sqrt{a^2+b^2}$

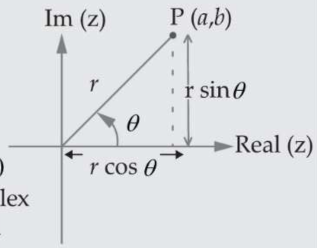
(ii) Angle θ made by OP with +ve direction of X-axis is called argument of z.



Polar Representation

Let $a=r\cos\theta$
$b=r\sin\theta$
where,
$r=|z|$
and $\theta=\arg(z)$
$\therefore z=a+ib=r(\cos\theta+i\sin\theta)$

The argument 'θ' of complex number z = a+ib is called principal argument of z if
$-\pi<\theta\le\pi$.



Square root of Complex Number

Let $x+iy=\sqrt{a+ib}$, squaring both sides, we get $(x+iy)^2=a+ib$ i.e. $x^2-y^2=a$, $2xy=b$ solving these equations, we get square root of z.

Multiplicative Inverse of complex Number

For a non-zero complex number z=a+ib $(a\neq0, b\neq0)$, there exists a complex number $\frac{a}{a^2+b^2}+\frac{-b}{a^2+b^2}i$ denoted by $\frac{1}{z}$ or z^{-1}, called multiplicative inverse of Z

Such that: $(a+ib)\left(\frac{a}{a^2+b^2}+i\frac{-b}{a^2+b^2}\right)=1+0i=1$

Solution of Quadratic Equation

- General form of quadratic equation in *x* is $ax^2+bx+c=0$,
  Where a,b,c ∈R & a ≠ 0
  The solutions of given quadratic equation are given by $x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
  $\because b^2-4ac<0$

Note: •A polynomial equation has atleast one root.
•A polynomial equation of degree n has n roots.

Important Questions

Multiple Choice questions-

Question 1. Let z_1 and z_2 be two roots of the equation $z^2 + az + b = 0$, z being complex. Further assume that the origin, z_1 and z_2 form an equilateral triangle. Then

(a) $a^2 = b$

(b) $a^2 = 2b$

(c) $a^2 = 3b$

(d) $a^2 = 4b$

Question 2. The value of i^i is

(a) 0

(b) $e^{-\pi}$

(c) $2e^{-\pi/2}$

(d) $e^{-\pi/2}$

Question 3. The value of $\sqrt{(-25)} + 3\sqrt{(-4)} + 2\sqrt{(-9)}$ is

(a) 13 i

(b) -13 i

(c) 17 i

(d) -17 i

So, $\sqrt{(-25)} + 3\sqrt{(-4)} + 2\sqrt{(-9)} = 17\,i$

Question 4. If the cube roots of unity are 1, ω and ω^2, then the value of $(1 + \omega / \omega^2)^3$ is

(a) 1

(b) -1

(c) ω

(d) ω^2

Question 5. If $\{(1 + i)/(1 – i)\}^n = 1$ then the least value of n is

(a) 1

(b) 2

(c) 3

(d) 4

Question 6. The value of $[i^{19} + (1/i)^{25}]^2$ is

(a) -1

(b) -2

(c) -3

(d) -4

Question 7. If z and w be two complex numbers such that |z| ≤ 1, |w| ≤ 1 and |z + iw| = |z – iw| = 2, then z equals {w is congugate of w}

(a) 1 or i

(b) i or – i

(c) 1 or – 1

(d) i or – 1

Question 8. The value of $\{-\sqrt{(-1)}\}^{4n+3}$, n ∈ N is

(a) i

(b) -i

(c) 1

(d) -1

Question 9. Find real θ such that (3 + 2i × sin θ)/(1 – 2i × sin θ) is real

(a) π

(b) nπ

(c) nπ/2

(d) 2nπ

Question 10. If i = √(-1) then $4 + 5(-1/2 + i\sqrt{3}/2)^{334} + 3(-1/2 + i\sqrt{3}/2)^{365}$ is equals to

(a) 1 – i√3

(b) -1 + i√3

(c) i√3

(d) -i√3

Very Short Questions:

Evaluate i^{-39}

1. Solved the quadratic equation $x^2 + x\frac{1}{\sqrt{2}} = 0$

2. If $\left(\frac{1+i}{1-i}\right)^m = 1$, then find the least positive integral value of m.

3. Evaluate $(1+ i)^4$

4. Find the modulus of $\frac{1+i}{1-i} - \frac{1-i}{1+1}$
5. Express in the form of a + ib. $(1+3i)^{-1}$
6. Explain the fallacy in -1 = i. i. = $\sqrt{-1}.\sqrt{-1} = \sqrt{-1(-1)} = \sqrt{1} = 1.$
7. Find the conjugate of $\frac{1}{2-3i}$
8. Find the conjugate of – 3i – 5.
9. Let $z_1 = 2 - i$, $z_2 = -2 + i$ Find Re $\left(\frac{z_1 z_2}{z_1}\right)$

Short Questions:

1. If x + iy = $\frac{a+ib}{a-ib}$ Prove that $x^2 + y^2 = 1$
2. Find real θ such that $\frac{3+2i\ sin\ \theta}{1-2i\ sin\ \theta}$ is purely real.
3. Find the modulus of $\frac{(1+i)(2+i)}{3+i}$
4. If |a + ib|= 1 then Show that $\frac{1+b+ai}{1+b-a} = b + ai$
5. If x – iy = $\sqrt{\frac{a-ib}{c-id}}$ Prove that $(x^2 + y^2)^2 = \frac{a^2+b^2}{c^2+d^2}$

Long Questions:

1. If z = x + i y and w = $\frac{1-i^2}{Z-i}$ Show that |w| = 1 ⇒ z is purely real.
2. Convert into polar form $\frac{-16}{1+i\sqrt{3}}$
3. Find two numbers such that their sum is 6 and the product is 14.
4. Convert into polar form $z = \frac{i-1}{\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}}$
5. If α and β are different complex number with |β| = 1 Then find $\left|\frac{\beta-\alpha}{1-\alpha\beta}\right|$

Assertion Reason Questions:

1. In each of the following questions, a statement of Assertion is given followed by a corresponding statement of Reason just below it. Of the statements, mark the correct answer as.

   Assertion (A): If $i = \sqrt{-1}$, then $i^{4k} = 1$, $i^{4k+1} = i$, $i^{4k+2} = -1$ and $i^{4k+3} = -i$.

   Reason (R) : $i^{4k} + i^{4k+1} + i^{4k+2} + i^{4k+3} = 1$.

   (i) Both assertion and reason are true and reason is the correct explanation of assertion.

(ii) Both assertion and reason are true but reason is not the correct explanation of assertion.

(iii) Assertion is true but reason is false.

(iv) Assertion is false but reason is true.

2. In each of the following questions, a statement of Assertion is given followed by a corresponding statement of Reason just below it. Of the statements, mark the correct answer as.

Assertion (A): Simplest form of i^{-35} is $-i$.

Reason (R) : Additive inverse of $(1 - i)$ is equal to $-1 + i$.

(i) Both assertion and reason are true and reason is the correct explanation of assertion.

(ii) Both assertion and reason are true but reason is not the correct explanation of assertion.

(iii) Assertion is true but reason is false.

(iv) Assertion is false but reason is true.

Answer Key:

MCQ

1. (c) $a^2 = 3b$

2. (d) $e^{-\pi/2}$

3. (c) 17 i

4. (b) -1

5. (d) 4

6. (d) -4

7. (c) 1 or – 1

8. (a) i

9. (b) nπ

10. (c) i√3

Very Short Answer:

1.

$$i^{-39} = \frac{1}{i^{39}} = \frac{1}{(i^4)^9 . i^3}$$

$= \frac{1}{1 \times (-i)}$ $\left[\because i^4 = 1, \; i^3 = -i\right.$

$= \frac{1}{-i} \times \frac{i}{i}$

$= \frac{i}{-i^2} = \frac{i}{-(-1)} = i$ $\left[\because i^2 = -1\right.$

2.

$\frac{x^2}{1} + \frac{x}{1} + \frac{1}{\sqrt{2}} = 0$

$\frac{\sqrt{2}x^2 + \sqrt{2}x + 1}{\sqrt{2}} = \frac{0}{1}$

$\sqrt{2}x^2 + \sqrt{2}x + 1 = 0$

$x = \frac{-b \pm \sqrt{D}}{2a}$

$= \frac{-\sqrt{2} \pm \sqrt{2 - 4\sqrt{2}}}{2 \times \sqrt{2}}$

$= \frac{-\sqrt{2} \pm \sqrt{2}\sqrt{1 - 2\sqrt{2}}}{2\sqrt{2}}$

$= \frac{-1 \pm \sqrt{2\sqrt{2} - 1}\, i}{2}$

3.

$\left(\frac{1+i}{1-i}\right)^m = 1$

$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^m = 1$

$\left(\frac{1 + i^2 + 2i}{1 - i^2}\right)^m = 1$

$\left(\frac{\cancel{1} - \cancel{1} + 2i}{2}\right)^m = 1$ $\left[\because i^2 = -1\right.$

$i^m = 1$

m=4

4.

$(1+i)^4 = \left[(1+i)^2\right]^2$

$=\left(1+i^2+2i\right)^2$

$=\left(1-1+2i\right)^2$

$=\left(2i\right)^2=4i^2$

$=4\left(-1\right)=-4$

5.

Let z = $\frac{1+i}{1-i}-\frac{1-i}{1+i}$

$=\frac{(1+i)^2-(1-i)^2}{(1-i)(1+i)}$

$=\frac{4i}{2}$

$=2i$

$z=0+2i$

$|z|=\sqrt{(0)^2+(2)^2}$

$=2$

6.

$(1+3i)^{-1}=\frac{1}{1+3i}\times\frac{1-3i}{1-3i}$

$=\frac{1-3i}{(1)^2-(3i)^2}$

$=\frac{1-3i}{1-9i^2}$

$=\frac{1-3i}{1+9}\quad\left[i^2=-1\right.$

$=\frac{1-3i}{10}$

$=\frac{1}{10}-\frac{3i}{10}$

7.

$1=\sqrt{1}=\sqrt{(-1)(-1)}$ is okay but

$\sqrt{(-1)(-1)}=\sqrt{-1}\sqrt{-1}$ is wrong.

8.

Let z = $\frac{1}{2-3i}$

$$z = \frac{1}{2-3i} \times \frac{2+3i}{2+3i} i$$

$$= \frac{2+3i}{(2)^2 - (3i)^2}$$

$$= \frac{2+3i}{4+9}$$

$$= \frac{2+3i}{13}$$

$$z = \frac{2}{13} + \frac{3}{13} i$$

$$\bar{z} = \frac{2}{13} - \frac{3}{13}$$

9. Let z = 3i – 5

$$\bar{z} = 3i - 5$$

10. z_1 z_2 = (2 – i)(-2 + i)

$$= -4 + 2i + 2i - i^2$$

$$= -4 + 4i + 1$$

$$= 4i - 3$$

$$\overline{z_1} = 2 + i$$

$$\frac{z_1 z_2}{\overline{z}_1} = \frac{4i-3}{2+i} \times \frac{2-i}{2-i}$$

$$= \frac{8i - 6 - 4i^2 + 3i}{4 - i^2}$$

$$= \frac{11i - 2}{5}$$

$$\frac{z_1 z_2}{z_1} = \frac{11}{5} i - \frac{2}{5}$$

$$\text{Re}\left(\frac{z_1 z_2}{\overline{z}_1}\right) = -\frac{2}{5}$$

Short Answer:

1.

$$x + iy = \frac{a+ib}{a-ib} \quad \text{(i) (Given)}$$

taking conjugate both side

$$x - iy = \frac{a-ib}{a+ib} \quad \text{(ii)}$$

$$\text{(i)} \times \text{(ii)}$$

$$(x+iy)(x-iy)=\left(\frac{a+ib}{a-ib}\right)\times\left(\frac{a-ib}{a+ib}\right)$$

$$(x)^2-(iy)^2=1$$

$x^2 + y^2 = 1$

$[i^2 = -1]$

2.

$$\frac{3+2i\ Sin\theta}{1-2i\ Sin\theta}=\frac{3+2i\ Sin\theta}{1-2i\ Sin\theta}\times\frac{1+2i\ Sin\theta}{1+2i\ Sin\theta}$$

$$=\frac{3+6i\ Sin\theta+2i\ Sin\theta-4Sin^2\theta}{1+4Sin^2\theta}$$

$$=\frac{3-4\ Sin^2\theta}{1+4\ Sin^2\theta}+\frac{8i\ Sin\theta}{1+4\ Sin^2\theta}$$

For purely real

Im (z) = 0

$$\frac{8Sin\theta}{1+4Sin^2\theta}=0$$

$$Sin\theta=0$$

$$\theta=n\pi$$

3.

$$\left|\frac{(1+i)(2+i)}{3+i}\right|=\frac{|(1+i)||2+i|}{|3+i|}$$

$$=\frac{\left(\sqrt{1^2+1^2}\right)\left(\sqrt{4+1}\right)}{\sqrt{(3)^2+(1)^2}}$$

$$=\frac{\left(\sqrt{2}\right)\left(\sqrt{5}\right)}{\sqrt{10}}$$

$$=\frac{\sqrt{2}\times\sqrt{5}}{\sqrt{2}\times\sqrt{5}}$$

$$=1$$

4.

$$|a+ib|=1$$

$$\sqrt{a^2+b^2}=1$$

$$a^2+b^2=1$$

$$\frac{1+b+ai}{1+b-ai}=\frac{(1+b)+ai}{(1+b)-ai}\times\frac{(1+b)+ai}{(1+b)+ai}$$

$$=\frac{(1+b)^2+(ai)^2+2(1+b)(ai)}{(1+b)^2-(ai)^2}$$

$$=\frac{1+b^2+2b-a^2+2ai+2abc}{1+b^2+2a-a^2}$$

$$=\frac{(a^2+b^2)+b^2+2b-a^2+2ai+2abi}{(a^2+b^2)+b^2+2b-a^2}$$

$$=\frac{2b^2+2b+2ai+2abi}{2b^2+2b}$$

$$=\frac{b^2+b+ai+abi}{b^2+b}$$

$$=\frac{b(b+1)+ai(b+1)}{b(b+1)}$$

$$=b+ai$$

5.

$$x-iy=\sqrt{\frac{a-ib}{c-id}} \quad (1)\ (\text{Given})$$

Taking conjugate both side

$$x+iy=\sqrt{\frac{a+ib}{c+id}} \quad (\text{ii})$$

(i) × (ii)

$$(x-iy)\times(x+iy)=\sqrt{\frac{a-ib}{c-id}}\times\sqrt{\frac{a+ib}{c+id}}$$

$$(x)^2-(iy)^2=\sqrt{\frac{(a)^2-(ib)^2}{(c)^2-(id)^2}}$$

$$x^2+y^2=\sqrt{\frac{a^2+b^2}{c^2+d^2}}$$

squaring both side

$$(x^2+y^2)^2=\frac{a^2+b^2}{c^2+d^2}$$

Long Answer:

1.

$$w=\frac{1-iz}{z-i}$$

$$=\frac{1-i(x+iy)}{x+iy-i}$$

$$=\frac{1-ix-i^2y}{x+i(y-1)}$$

$$=\frac{(1+y)-ix}{x+i(y-1)}$$

$\therefore\ |w|=1$

$$\Rightarrow\left|\frac{(1+y)-ix}{x+i(y-1)}\right|=1$$

$$\frac{|(1+y)-ix|}{|x+i(y-1)|}=1$$

$$\frac{\sqrt{(1+y)^2+(-x)^2}}{\sqrt{x^2+(y-1)^2}}=1$$

$1+y^2+2y+x^2=x^2+y^2+1-2y$

$4y=0$

$y=0$

$\therefore\ z=x+i$

is purely real

2.

$$\frac{-16}{1+i\sqrt{3}}=\frac{-16}{1+i\sqrt{3}}\times\frac{1-i\sqrt{3}}{1-i\sqrt{3}}$$

$$=\frac{-16\left(1-i\sqrt{3}\right)}{(1)^2-\left(i\sqrt{3}\right)^2}$$

$$=\frac{-16\left(1-i\sqrt{3}\right)}{1+3}$$

$=-4\left(1-i\sqrt{3}\right)$

$z=-4+i4\sqrt{3}$

$r=\ |z|=\sqrt{(-4)^2+\left(4\sqrt{3}\right)^2}$

$=\sqrt{16+48}$

$=\sqrt{64}$

$=8$

Let α be the acute $\angle$S

$$\tan\alpha=\left|\frac{\cancel{4}\sqrt{3}}{-\cancel{4}}\right|$$

$\tan\alpha = \tan\frac{\pi}{3}$

$\alpha = \pi/3$

Since Re (z) < o, and Im (z) > o

$\theta = \pi - \alpha$

$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}$

$z = 8\left(Cos\ \frac{2\pi}{3} + i\ \text{Sin}\ \frac{2\pi}{3}\right)$

3.

Let x and y be the no.

x + y = 6

xy = 14

$x^2 - 6x + 14 = 0$

$D = -20$

$x = \frac{-(-6) \pm \sqrt{-20}}{2 \times 1}$

$= \frac{6 \pm 2\sqrt{5}\ \text{i}}{2}$

$= 3 \pm \sqrt{5}\ \text{i}$

$\text{x} = 3 + \sqrt{5}\ \text{i}$

$y = 6 - \left(3 + \sqrt{5}\ \text{i}\right)$

$= 3 - \sqrt{5}\ \text{i}$

when $\text{x} = 3 - \sqrt{5}\ \text{i}$

$y = 6 - \left(3 - \sqrt{5}\ \text{i}\right)$

$= 3 + \sqrt{5}\ \text{i}$

4.

$z = \frac{i-1}{\frac{1}{2} + \frac{\sqrt{3}}{2}i}$

$= \frac{2(i-1)}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i}$

$z = \frac{\sqrt{3}-1}{2} + \frac{\sqrt{3}+1}{2}i$

$$r=|z|=\left(\frac{\sqrt{3}-1}{2}\right)^2+\left(\frac{\sqrt{3}+1}{2}\right)^2$$

$$r=2$$

Let α be the acule $\angle$s

$$\tan\ \alpha=\left|\frac{\frac{\sqrt{3}+1}{2}}{\frac{\sqrt{3}-1}{2}}\right|$$

$$=\left|\frac{\sqrt{3}\left(1+\frac{1}{\sqrt{3}}\right)}{\sqrt{3}\left(1-\frac{1}{\sqrt{3}}\right)}\right|$$

$$=\left|\frac{\tan\frac{\pi}{4}+\tan\frac{\pi}{6}}{1-\tan\frac{\pi}{4}\tan\frac{\pi}{6}}\right|$$

$$\tan\alpha=\left|\tan\left(\frac{\pi}{4}+\frac{\pi}{6}\right)\right|$$

$$\alpha=\frac{\pi}{4}+\frac{\pi}{6}=\frac{5\pi}{12}$$

$$z=2\left(Cos\frac{5\pi}{12}+i\ \text{Sin}\ \frac{5\pi}{12}\right)$$

5.

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|^2=\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\overline{\frac{\beta-\alpha}{1-\bar{\alpha}\beta}}\right) \quad \left[\because |z|^2=z\bar{z}\right.$$

$$=\left(\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right)\left(\frac{\bar{\beta}-\bar{\alpha}}{1-\alpha\bar{\beta}}\right)$$

$$=\left(\frac{\beta\bar{\beta}-\beta\bar{\alpha}-\alpha\bar{\beta}+\alpha\bar{\alpha}}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+\alpha\bar{\alpha}\beta\bar{\beta}}\right)$$

$$=\left(\frac{|\beta|^2-\beta\bar{\alpha}-\alpha\bar{\beta}+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2|\beta|^2}\right)$$

$$=\left(\frac{1-\beta\bar{\alpha}-\alpha\bar{\beta}+|\alpha|^2}{1-\alpha\bar{\beta}-\bar{\alpha}\beta+|\alpha|^2}\right) \quad [\because |\beta|=1$$

$$=1$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=\sqrt{1}$$

$$\left|\frac{\beta-\alpha}{1-\bar{\alpha}\beta}\right|=1$$

Assertion Reason Answer:

1. (iii) Assertion is true but reason is false.
2. (iv) Assertion is false but reason is true.